valence electrons per unit volume for n_Q in (12.100) yields only order-of-magnitude estimates for λ_L. In passing we note that in high-temperature (cupric oxide) superconductors penetration depths are found to be an order of magnitude smaller than in conventional superconductors.

Measurements of λ_L, especially its temperature dependence, can be accomplished by incorporating the superconducting specimen into a resonant circuit and studying the shift in resonant frequency with change in temperature. In circumstances in which λ_L is small compared to both the wavelength λ associated with the resonant circuit and the sample size, a simple calculation (Problem 12.20) paralleling Section 8.1 leads to a purely reactive surface impedance,

$$Z_s \approx -i\frac{8\pi^2}{c}\frac{\lambda_L}{\lambda} \text{ (Gaussian units)} \qquad \text{or} \qquad Z_s \approx -i\frac{2\pi\lambda_L}{\lambda} Z_0 \text{ (SI units)}$$

With our convention about time dependence ($e^{-i\omega t}$), the impedance is inductive, corresponding to an inductance per unit area, $L = \mu_0\lambda_L$ (SI units).

Our sketch of the simple London theory addresses only the Meissner effect, and not all of it. The magnetic and thermodynamic properties of superconductors, the physical size of the coherent state (coherence length ξ), and many other features are fully addressed only by the microscopic quantum-mechanical theory. The reader wishing to learn more about superconductivity may consult *Ashcroft and Mermin* or *Kittel* and the numerous references cited there. An alternative, perhaps more physical, approach (also by F. London) to the London equations is addressed in Problem 12.21.

12.10 *Canonical and Symmetric Stress Tensors; Conservation Laws*

A. *Generalization of the Hamiltonian: Canonical Stress Tensor*

In particle mechanics the transition to the Hamiltonian formulation and conservation of energy is made by first defining the canonical momentum variables

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and then introducing the Hamiltonian

$$H = \sum_i p_i\dot{q}_i - L \tag{12.101}$$

It can then be shown that $dH/dt = 0$ provided $\partial L/\partial t = 0$. For fields we anticipate having a Hamiltonian density whose volume integral over three-dimensional space is the Hamiltonian. The Lorentz transformation properties of $\mathcal{H}$ can be guessed as follows. Since the energy of a particle is the time component of a 4-vector, the Hamiltonian should transform in the same way. Since $H = \int \mathcal{H}\, d^3x$, and the invariant 4-volume element is $d^4x = d^3x\, dx_0$, it is necessary that the Hamiltonian density $\mathcal{H}$ transform as the time-time component of a second-rank tensor. If the Lagrangian density for some fields is a function of the field variables

$\phi_k(x)$, $\partial^\alpha \phi_k(x)$, $k = 1, 2, \ldots, n$, the Hamiltonian density is defined in analogy with (12.101) as

$$\mathcal{H} = \sum_k \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \phi_k}{\partial t}\right)} \frac{\partial \phi_k}{\partial t} - \mathcal{L} \tag{12.102}$$

The first factor in the sum is the field momentum canonically conjugate to $\phi_k(x)$ and $\partial\phi_k/\partial t$ is equivalent to the velocity $\dot{q}_i$. The inferred Lorentz transformation properties of $\mathcal{H}$ suggest that the *covariant generalization of the Hamiltonian* density is the *canonical stress tensor*:

$$T^{\alpha\beta} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k - g^{\alpha\beta}\mathcal{L} \tag{12.103}$$

For the free electromagnetic field Lagrangian

$$\mathcal{L}_{em} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$$

the canonical stress tensor is

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}_{em}}{\partial(\partial_\alpha A^\lambda)} \partial^\beta A^\lambda - g^{\alpha\beta}\mathcal{L}_{em}$$

where a summation over λ is implied by the repeated index. With the help of (12.87) (but notice the placing of the indices!) we find

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\mu}F_{\mu\lambda}\partial^\beta A^\lambda - g^{\alpha\beta}\mathcal{L}_{em} \tag{12.104}$$

To elucidate the meaning of the tensor we exhibit some components. With $\mathcal{L} = (\mathbf{E}^2 - \mathbf{B}^2)/8\pi$ and (11.138) we find

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) + \frac{1}{4\pi}\boldsymbol{\nabla}\cdot(\Phi\mathbf{E}) \\ T^{0i} &= \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i + \frac{1}{4\pi}\boldsymbol{\nabla}\cdot(A_i\mathbf{E}) \\ T^{i0} &= \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i + \frac{1}{4\pi}\left[(\boldsymbol{\nabla}\times\Phi\mathbf{B})_i - \frac{\partial}{\partial x_0}(\Phi E_i)\right] \end{aligned} \tag{12.105}$$

In writing the second terms here we have made use of the free-field equations $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ and $\boldsymbol{\nabla}\times\mathbf{B} - \partial\mathbf{E}/\partial x_0 = 0$. If we suppose that the fields are localized in some finite region of space (and, because of the finite velocity of propagation, they always are), the integrals over all 3-space at fixed time in some inertial frame of the components T^{00} and T^{0i} can be interpreted, as in Chapter 6, as the total energy and c times the total momentum of the electromagnetic fields in that frame:

$$\begin{aligned} \int T^{00}\, d^3x &= \frac{1}{8\pi}\int (\mathbf{E}^2 + \mathbf{B}^2)\, d^3x = E_{\text{field}} \\ \int T^{0i}\, d^3x &= \frac{1}{4\pi}\int (\mathbf{E}\times\mathbf{B})_i\, d^3x = cP^i_{\text{field}} \end{aligned} \tag{12.106}$$

These are the usual (Gaussian units) expressions for the total energy and momentum of the fields, discussed in Section 6.7. We note that the components T^{00} and T^{0i} themselves differ from the standard definitions of energy density and momentum density by added divergences. Upon integration over all space, however, the added terms give no contribution, being transformed into surface integrals at infinity where all the fields and potentials are identically zero.

The connection of the time-time and time-space components of $T^{\alpha\beta}$ with the field energy and momentum densities suggests that there is a covariant generalization of the differential conservation law (6.108) of Poynting's theorem. This differential conservation statement is

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{12.107}$$

In proving (12.107) we treat the general situation described by the tensor (12.103) and the Euler–Lagrange equations (12.83). Consider

$$\begin{aligned}\partial_\alpha T^{\alpha\beta} &= \sum_k \partial_\alpha\left[\frac{\partial\mathscr{L}}{\partial(\partial_\alpha\phi_k)}\,\partial^\beta\phi_k\right] - \partial^\beta\mathscr{L} \\ &= \sum_k\left[\partial_\alpha\,\frac{\partial\mathscr{L}}{\partial(\partial_\alpha\phi_k)}\cdot\partial^\beta\phi_k + \frac{\partial\mathscr{L}}{\partial(\partial_\alpha\phi_k)}\,\partial_\alpha\partial^\beta\phi_k\right] - \partial^\beta\mathscr{L}\end{aligned}$$

By means of the equations of motion (12.83) the first term can be transformed so that

$$\partial_\alpha T^{\alpha\beta} = \sum_k\left[\frac{\partial\mathscr{L}}{\partial\phi_k}\,\partial^\beta\phi_k + \frac{\partial\mathscr{L}}{\partial(\partial_\alpha\phi_k)}\,\partial^\beta(\partial_\alpha\phi_k)\right] - \partial^\beta\mathscr{L}$$

Since $\mathscr{L} = \mathscr{L}(\phi_k, \partial^\alpha\phi_k)$, the square bracket, summed, is the expression for an implicit differentiation (chain rule). Hence

$$\partial_\alpha T^{\alpha\beta} = \partial^\beta\mathscr{L}(\phi_k, \partial^\alpha\phi_k) - \partial^\beta\mathscr{L} = 0$$

The conservation law or continuity equation (12.107) yields the conservation of total energy and momentum upon integration over all of 3-space at fixed time. Explicitly, we have

$$0 = \int \partial_\alpha T^{\alpha\beta}\,d^3x = \partial_0\int T^{0\beta}\,d^3x + \int \partial_i T^{i\beta}\,d^3x$$

If the fields are localized the second integral (a divergence) gives no contribution. Then with the identifications (12.106) we find

$$\frac{d}{dt}E_{\text{field}} = 0, \qquad \frac{d}{dt}\mathbf{P}_{\text{field}} = 0 \tag{12.108}$$

In this derivation of the conservation of energy (Poynting's theorem) and momentum and in the definitions (12.106) we have not exhibited manifest covariance. The results are valid for an observer at rest in the frame in which the fields are specified. But the question of transforming from one frame to another has not been addressed. With a covariant differential conservation law, $\partial_\alpha T^{\alpha\beta} = 0$, one expects that a covariant integral statement is also possible. The integrals in (12.106) do not appear to have the transformation properties of the components of a 4-vector. For source-free fields they do in fact transform properly (see Problem 12.18 and *Rohrlich*, Appendix A1-5), but in general do not. To avoid having electromagnetic energy and momentum defined separately in each inertial frame, without the customary connection between frames, one may construct explicitly covariant integral expressions for the electromagnetic energy and momentum, of which the forms (12.106) are special cases, valid in only one reference frame. This is discussed further in Chapter 16 in the context of the classical electromagnetic self-energy problem. (See *Rohrlich*, Section 4-9, for an explicitly covariant treatment of the conservation laws in integral form.)

B. Symmetric Stress Tensor

The canonical stress tensor $T^{\alpha\beta}$, while adequate so far, has a certain number of deficiencies. We have already seen that T^{00} and T^{0i} differ from the usual expressions for energy and momentum densities. Another drawback is its lack of symmetry—see T^{0i} and T^{i0} in (12.105). The question of symmetry arises when we consider the angular momentum of the field,

$$\mathbf{L}_{\text{field}} = \frac{1}{4\pi c} \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3x$$

The angular momentum density has a covariant generalization in terms of the third-rank tensor,

$$M^{\alpha\beta\gamma} = T^{\alpha\beta}x^\gamma - T^{\alpha\gamma}x^\beta \tag{12.109}$$

Then, just as (12.107) implies (12.108), so the vanishing of the 4-divergence

$$\partial_\alpha M^{\alpha\beta\gamma} = 0 \tag{12.110}$$

implies conservation of the total angular momentum of the field. Direct calculation of (12.110) gives

$$0 = (\partial_\alpha T^{\alpha\beta})x^\gamma + T^{\gamma\beta} - (\partial_\alpha T^{\alpha\gamma})x^\beta - T^{\beta\gamma}$$

With (12.107) eliminating the first and third terms, we see that conservation of angular momentum requires that $T^{\alpha\beta}$ be symmetric. Two final criticisms of $T^{\alpha\beta}$, (12.104), are that it involves the potentials explicitly, and so is not gauge invariant, and that its trace (T^α_α) is not zero, as required for zero-mass photons.

There is a general procedure for constructing a symmetric, traceless, gauge-invariant stress tensor $\Theta^{\alpha\beta}$ from the canonical stress tensor $T^{\alpha\beta}$ (see the references at the end of the chapter). For the electromagnetic $T^{\alpha\beta}$ of (12.104) we proceed directly. We substitute $\partial^\beta A^\lambda = -F^{\lambda\beta} + \partial^\lambda A^\beta$ and obtain

$$T^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] - \frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}\,\partial^\lambda A^\beta \tag{12.111}$$

The first terms in (12.111) are symmetric in α and β and gauge invariant. With the help of the source-free Maxwell equations, the last term can be written

$$\begin{aligned} T_D^{\alpha\beta} &\equiv -\frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}\,\partial^\lambda A^\beta = \frac{1}{4\pi}F^{\lambda\alpha}\,\partial_\lambda A^\beta \\ &= \frac{1}{4\pi}(F^{\lambda\alpha}\partial_\lambda A^\beta + A^\beta\partial_\lambda F^{\lambda\alpha}) \qquad (12.112) \\ &= \frac{1}{4\pi}\partial_\lambda(F^{\lambda\alpha}A^\beta) \end{aligned}$$

The tensor $T_D^{\alpha\beta}$ has the following easily verified properties:

$$\text{(i)} \qquad \partial_\alpha T_D^{\alpha\beta} = 0$$
$$\text{(ii)} \qquad \int T_D^{0\beta}\, d^3x = 0$$

Thus the differential conservation law (12.107) will hold for the difference $(T^{\alpha\beta} - T_D^{\alpha\beta})$ if it holds for $T^{\alpha\beta}$. Furthermore, the integral relations (12.106) for the total energy and momentum of the fields will also be valid in terms of the difference tensor. We are therefore free to *define the symmetric stress tensor* $\Theta^{\alpha\beta}$:

$$\Theta^{\alpha\beta} = T^{\alpha\beta} - T_D^{\alpha\beta}$$

or

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}\,g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}\right) \tag{12.113}$$

Explicit calculation gives the following components,

$$\begin{aligned}
\Theta^{00} &= \frac{1}{8\pi}\,(E^2 + B^2) \\
\Theta^{0i} &= \frac{1}{4\pi}\,(\mathbf{E} \times \mathbf{B})_i \\
\Theta^{ij} &= \frac{-1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\,\delta_{ij}(E^2 + B^2)\right]
\end{aligned} \tag{12.114}$$

The indices i and j refer to Cartesian components in 3-space. The tensor $\Theta^{\alpha\beta}$ can be written in schematic matrix form as

$$\Theta^{\alpha\beta} = \left(\begin{array}{c|c} u & c\mathbf{g} \\ \hline c\mathbf{g} & -T_{ij}^{(M)} \end{array}\right) \tag{12.115}$$

In (12.115) the time-time and time-space components are expressed as the energy and momentum densities (6.106) and (6.118), now in Gaussian units, while the space-space components (12.114) are seen to be just the negative of the Maxwell stress tensor (6.120) in Gaussian units, denoted here by $T_{ij}^{(M)}$ to avoid confusion with the canonical tensor $T^{\alpha\beta}$. The various other, covariant and mixed, forms of the stress tensor are

$$\Theta_{\alpha\beta} = \left(\begin{array}{c|c} u & -c\mathbf{g} \\ \hline -c\mathbf{g} & -T_{ij}^{(M)} \end{array}\right) \qquad \Theta^{\alpha}{}_{\beta} = \left(\begin{array}{c|c} u & -c\mathbf{g} \\ \hline c\mathbf{g} & T_{ij}^{(M)} \end{array}\right)$$

$$\Theta_{\alpha}{}^{\beta} = \left(\begin{array}{c|c} u & c\mathbf{g} \\ \hline -c\mathbf{g} & T_{ij}^{(M)} \end{array}\right)$$

The differential conservation law

$$\partial_\alpha \Theta^{\alpha\beta} = 0 \tag{12.116}$$

embodies Poynting's theorem and conservation of momentum for free fields. For example, with $\beta = 0$ we have

$$0 = \partial_\alpha \Theta^{\alpha 0} = \frac{1}{c}\left(\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S}\right)$$

where $\mathbf{S} = c^2\mathbf{g}$ is the Poynting vector. This is the source-free form of (6.108). Similarly, for $\beta = i$,

$$0 = \partial_\alpha \Theta^{\alpha i} = \frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}^{(M)}$$

a result equivalent to (6.121) in the absence of sources. The conservation of field angular momentum, defined through the tensor

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta \tag{12.117}$$

is assured by (12.116) and the symmetry of $\Theta^{\alpha\beta}$, as already discussed. There are evidently other conserved quantities in addition to energy, momentum, and angular momentum. The tensor $M^{0\beta\gamma}$ has three time-space components in addition to the space-space components that give the angular momentum density. These three components are a necessary adjunct of the covariant generalization of angular momentum. Their conservation is a statement on the center of mass motion (see Problem 12.19).

C. Conservation Laws for Electromagnetic Fields Interacting with Charged Particles

In the presence of external sources the Lagrangian for the Maxwell equations is (12.85). The symmetric stress tensor for the electromagnetic field retains its form (12.113), but the coupling to the source current makes its divergence nonvanishing. The calculation of the divergence is straightforward:

$$\begin{aligned}\partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{4\pi}\left[\partial^\mu(F_{\mu\lambda}F^{\lambda\beta}) + \frac{1}{4}\partial^\beta(F_{\mu\lambda}F^{\mu\lambda})\right] \\ &= \frac{1}{4\pi}\left[(\partial^\mu F_{\mu\lambda})F^{\lambda\beta} + F_{\mu\lambda}\,\partial^\mu F^{\lambda\beta} + \frac{1}{2}F_{\mu\lambda}\,\partial^\beta F^{\mu\lambda}\right]\end{aligned}$$

The first term can be transformed by means of the inhomogeneous Maxwell equations (12.89). Transferring this term to the left-hand side, we have

$$\partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = \frac{1}{8\pi}F_{\mu\lambda}(\partial^\mu F^{\lambda\beta} + \underline{\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda}})$$

The reason for the peculiar grouping of terms is that the underlined sum can be replaced, by virtue of the homogeneous Maxwell equation $(\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda} + \partial^\lambda F^{\beta\mu} = 0)$, by $-\partial^\lambda F^{\beta\mu} = +\partial^\lambda F^{\mu\beta}$. Thus we obtain

$$\partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = \frac{1}{8\pi}F_{\mu\lambda}(\partial^\mu F^{\lambda\beta} + \partial^\lambda F^{\mu\beta})$$

But the right-hand side is now the contraction (in μ and λ) of one symmetric and one antisymmetric factor. The result is therefore zero. The divergence of the stress tensor is thus

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{-1}{c} F^{\beta\lambda} J_\lambda \tag{12.118}$$

The time and space components of this equation are

$$\frac{1}{c}\left(\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S}\right) = -\frac{1}{c} \mathbf{J} \cdot \mathbf{E} \tag{12.119}$$

and

$$\frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}^{(M)} = -\left[\rho E_i + \frac{1}{c} (\mathbf{J} \times \mathbf{B})_i\right] \tag{12.120}$$

These are just the conservation of energy and momentum equations of Chapter 6 for electromagnetic fields interacting with sources described by $J^\alpha = (c\rho, \mathbf{J})$. The negative of the 4-vector on the right-hand side of (12.118) is called the *Lorentz force density*,

$$f^\beta \equiv \frac{1}{c} F^{\beta\lambda} J_\lambda = \left(\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho\mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}\right) \tag{12.121}$$

If the sources are a number of charged particles, the volume integral of f^β leads through the Lorentz force equation (12.1) to the time rate of change of the sum of the energies or the momenta of all particles:

$$\int f^\beta \, d^3x = \frac{dP^\beta_{\text{particles}}}{dt}$$

With the qualification expressed at the end of Section 12.10.A concerning covariance, the integral over 3-space at fixed time of the left-hand side of (12.118) is the time rate of change of the total energy or momentum of the field. We therefore have the conservation of 4-momentum for the combined system of particles and fields:

$$\int d^3x (\partial_\alpha \Theta^{\alpha\beta} + f^\beta) = \frac{d}{dt} (P^\beta_{\text{field}} + P^\beta_{\text{particles}}) = 0 \tag{12.122}$$

The discussion above focused on the electromagnetic field, with charged particles only mentioned as the sources of the 4-current density. A more equitable treatment of a combined system of particles and fields involves a Lagrangian having three terms, a free-field Lagrangian, a free-particle Lagrangian, and an interaction Lagrangian that involves both field and particle degrees of freedom. Variation of the action integral with respect to the particle coordinates leads to the Lorentz force equation, just as in Section 12.1, while variation of the field "coordinates" gives the Maxwell equations, as in Section 12.7. However, when self-energy and radiation reaction effects are included, the treatment is not quite so straightforward. References to these aspects are given at the end of the chapter.

Mention should also be made of the action-at-a-distance approach associated with the names of Schwarzschild, Tetrode, and Fokker. The emphasis is on the charged particles and an invariant action principle is postulated with the interaction term involving integrals over the world lines of all the particles. The idea of electromagnetic fields and the Maxwell equations is secondary. This approach is the basis of the Wheeler–Feynman absorber theory of radiation.*

12.11 *Solution of the Wave Equation in Covariant Form; Invariant Green Functions*

The electromagnetic fields $F^{\alpha\beta}$ arising from an external source $J^\alpha(x)$ satisfy the inhomogeneous Maxwell equations

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

With the definition of the fields in terms of the potentials this becomes

$$\Box A^\beta - \partial^\beta(\partial_\alpha A^\alpha) = \frac{4\pi}{c} J^\beta$$

If the potentials satisfy the Lorenz condition, $\partial_\alpha A^\alpha = 0$, they are then solutions of the four-dimensional wave equation,

$$\Box A^\beta = \frac{4\pi}{c} J^\beta(x) \tag{12.123}$$

The solution of (12.123) can be accomplished by finding a Green function $D(x, x')$ for the equation

$$\Box_x D(x, x') = \delta^{(4)}(x - x') \tag{12.124}$$

where $\delta^{(4)}(x - x') = \delta(x_0 - x_0')\,\delta(\mathbf{x} - \mathbf{x}')$ is a four-dimensional delta function. In the absence of boundary surfaces, the Green function can depend only on the 4-vector difference $z^\alpha = x^\alpha - x'^\alpha$. Thus $D(x, x') = D(x - x') = D(z)$ and (12.124) becomes

$$\Box_z D(z) = \delta^{(4)}(z)$$

We use Fourier integrals to transform from coordinate to wave number space. The Fourier transform $\tilde{D}(k)$ of the Green function is defined by

$$D(z) = \frac{1}{(2\pi)^4} \int d^4k\, \tilde{D}(k) e^{-ik\cdot z} \tag{12.125}$$

where $k \cdot z = k_0 z_0 - \mathbf{k} \cdot \mathbf{z}$. With the representation of the delta function being

$$\delta^{(4)}(z) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik\cdot z} \tag{12.126}$$

one finds that the k-space Green function is

$$\tilde{D}(k) = -\frac{1}{k \cdot k} \tag{12.127}$$

*J. A. Wheeler and R. P. Feynman, *Rev. Mod. Phys.* **21**, 425 (1949).